## TOPOLOGICAL $\sigma$ -MODEL, HAMILTONIAN DYNAMICS AND LOOP SPACE LEFSCHETZ NUMBER

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We use path integral methods and topological quantum field theory techniques to investigate a generic classical Hamiltonian system. In particular, we show that Floer's instanton equation is related to a functional Euler character in the quantum cohomology defined by the topological nonlinear  $\sigma$ -model. This relation is an infinite dimensional analog of the relation between Poincaré-Hopf and Gauss-Bonnet-Chern formulæ in classical Morse theory, and can also be viewed as a loop space generalization of the Lefschetz fixed point theorem. By applying localization techniques to path integrals we then show that for a Kähler manifold our functional Euler character coincides with the Euler character determined by the finite dimensional de Rham cohomology of the phase space. Our results are consistent with the Arnold conjecture which estimates periodic solutions to classical Hamilton's equations in terms of de Rham cohomology of the phase space.

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The methods of quantum field theory were originally developed to understand particle physics, but have since proven useful also in statistical physics. Recently there have been various indications that these methods might be successfully applied even in the study of *classical* Hamiltonian dynamics. One of the intriguing open problems there is the Arnold conjecture [1], [2] which states that on a compact phase space the number of periodic solutions to Hamilton's equations is bounded from below by the sum of Betti numbers on the phase space.

In the present Letter we shall argue that topological quantum field theories [3] and path integral localization techniques [4] are indeed potentially effective tools in the study of classical dynamical systems. Furthermore, by applying such methods to investigate classical structures one might also gain valuable insight to the nonperturbative structure of quantum dynamics.

We shall consider Hamilton's equations on a phase space which is a compact symplectic manifold X with local coordinates  $\phi^a$ . We are interested in T-periodic trajectories that solve Hamilton's equations, *i.e.* are critical points of the classical action

$$S_{\rm cl} = \int_0^T d\tau (\vartheta_a \dot{\phi}^a - H(\phi, \tau)) \tag{1}$$

Here  $\vartheta_a$  are components of the symplectic potential corresponding to the symplectic two-form  $\omega = d\vartheta$ . We assume that the Hamiltonian depends *explicitly* on time  $\tau$  in a T-periodic manner  $H(\phi,0) = H(\phi,T)$ , so that energy is not necessarily conserved. Hamilton's equations are

$$\dot{\phi}^a - \omega^{ab} \partial_b H(\phi; \tau) = \dot{\phi}^a - \mathcal{X}_H^a = 0 \tag{2}$$

with T-periodic boundary condition  $\phi(0) = \phi(T)$ . We are particularly interested in the case where the periodic solutions are nondegenerate.

When energy is conserved so that H does not have explicit dependence on  $\tau$  each critical point  $dH(\phi) = 0$  of H generates trivially a T-periodic trajectory. According to the classical Morse theory the number of these critical points is bounded from below by the sum of Betti numbers on X and consequently Arnold's conjecture is valid:

$$\#\{ T - \text{periodic trajectories } \} \ge \sum B_k = \sum \dim H^k(X, R)$$
 (3)

But if H depends explicitly on time so that energy is *not* conserved, the critical points of H do not solve (2) and the methods of finite dimensional Morse theory are no longer applicable. Instead, we need to estimate directly the quantity

#{ 
$$T$$
-periodic trajectories } = 
$$\int_{\phi(T)=\phi(0)} [d\phi] \delta(\dot{\phi}^a - \mathcal{X}_H^a) \left| \det ||\delta_a^b \partial_t + \partial_a \mathcal{X}_H^b|| \right|$$
(4)

and for this we need an infinite dimensional generalization of Morse theory. Unfortunately this is not very easy. There is no minimum for (1), and periodic solutions of (2) are saddle points of (1) with an infinite Morse index<sup>1</sup>. Due to such difficulties,

In the following we shall define all determinants using  $\zeta$ -function regularization.

for explicitly  $\tau$ -dependent Hamiltonians the conjecture has only been proven in certain special cases [2].

An important special case is governed by the Lefschetz fixed point theorem [5]: We first estimate the integral in (4) by

$$\geq \left| \int_{\phi(T)=\phi(0)} [d\phi] \delta(\dot{\phi}^a - \mathcal{X}_H^a) \det ||\delta_a^b \partial_t + \partial_a \mathcal{X}_H^b|| \right| = \left| \sum_{\delta S_{PBC}=0} sign \left\{ \det ||\delta_a^b \partial_t + \partial_a \mathcal{X}_H^b|| \right\} \right|$$
(5)

where on the r.h.s. we identify an infinite dimensional version of the sum that appears in finite dimensional Poincaré-Hopf theorem with S as a Morse function. If we introduce a commuting variable  $p_a$  and two anticommuting variables  $c^a$ ,  $\bar{c}_a$ , we can write the integral in (5) as

$$\int [d\phi][dp][dc][d\bar{c}] \exp\{i \int_{a}^{T} p_a \dot{\phi}^a - \bar{c}_a \dot{c}^a - p_a \mathcal{X}_H^a - \bar{c}_a \partial_b \mathcal{X}_H^a c^b\}$$
 (6)

Furthermore, if we define the following nilpotent BRST operator

$$Q = c^a \frac{\partial}{\partial \phi^a} + p_a \frac{\partial}{\partial \bar{c}_a} \tag{7}$$

we find that this integral can be represented in the topological form

$$= \int [d\phi][dp][dc][d\bar{c}] \exp\{i \int_{0}^{T} p_a \dot{\phi}^a - \bar{c}_a \dot{c}^a - Q(\mathcal{X}_H^a \bar{c}_a)\}$$
 (8)

According to standard BRST arguments, this integral depends only on the cohomology of the BRST operator Q. In particular we can replace Q by its canonical conjugation  $Q \to e^{-\theta}Qe^{\theta}$  where we select  $\theta = -\Gamma^c_{ab}c^a\bar{c}_c\pi^b$ , with  $\Gamma^c_{ab}$  the connection for some metric  $g_{ab}$  on the phase space X and  $\pi^a p_b = \delta^a_b$ . Furthermore, if we generalize

$$\mathcal{X}_H^a \bar{c}_a \rightarrow (1-\lambda)\mathcal{X}_H^a \bar{c}_a + \lambda g^{ab} p_a \bar{c}_b$$

where  $\lambda \in [0, 1]$  is a parameter, BRST invariance implies that the ensuing path integral (8) is independent of  $\lambda$ . Selecting  $\lambda = 1$  we then conclude that (8) localizes to the Euler characteristic of the phase space X [3], [6]. Consequently

#{ 
$$T$$
-periodic trajectories }  $\geq \left| \int_X \operatorname{Pf}\left[\frac{1}{2}R^a{}_{bcd}c^cc^d\right] \right| = \left| \sum_k (-)^k H^k(X,R) \right|$  (9)

and for manifolds with vanishing odd Betti numbers i.e.  $H_{2k+1} = 0$  this establishes the validity of Arnold's conjecture. This result can be viewed as a path integral proof of the Lefschetz fixed point theorem [5]: If  $F: X \to X$  is a smooth map which is homotopic to the identity and admits only isolated fixed points  $F[\phi] = \phi$ , according

to this theorem the number of these fixed points is bounded from below by the Euler character of X. In the present context, Hamilton's equations determine such a smooth map  $F: X \to X$  by  $F[\phi] \equiv F[\phi(t=0)] \to \phi(T)$  and the time evolution determined by Hamilton's equations ensures that this map is homotopic to the identity. The fixed point condition  $F[\phi] = \phi$  then selects T-periodic classical trajectories  $\phi(0) = \phi(T)$ .

In spite of very extensive work [2] in the general case when the odd Betti-numbers are non-vanishing the Arnold conjecture remains unproven. However, recent attempts have led to interesting approaches, including topological nonlinear  $\sigma$ -models. In the approach originating from Floer [7, 2] one starts by defining a gradient flow in the space of closed loops  $\phi(0) = \phi(T)$ 

$$\frac{\partial \phi^a}{\partial \sigma} = -g^{ab} \frac{\delta S_{\rm cl}}{\delta \phi^b} \tag{10}$$

where  $g_{ab}$  is a Riemannian metric on X. Using this metric and the symplectic two-form  $\omega_{ab}$  we set

$$I^a{}_b = g^{ac}\omega_{cb}$$

Since  $I^a{}_c I^c{}_b = -\delta^a{}_b$  this defines an almost complex structure on the manifold X and (10) becomes

$$\partial_{\sigma}\phi^{a} + I^{a}{}_{b}\partial_{\tau}\phi^{b} = \partial^{a}H(\phi,\tau). \tag{11}$$

This is Floer's instanton equation, defined on a cylinder  $S^1 \times R$  with local coordinates  $\tau \in [0, T]$  and  $\sigma \in (-\infty, \infty)$ . It describes the flow of loops  $\phi(\tau)$  on X, and the bounded orbits in  $\sigma$  tend asymptotically to the periodic solutions of Hamilton's equation (2). Using (11) Floer constructs a complex with the solutions to (2) being the vertices and the trajectories (11), so-called pseudo-holomorphic instantons, connecting them as the edges. He proves the important result that the cohomology of this complex is independent of the Hamiltonian  $H(\phi, \tau)$ . Subsequently Witten [8] found that this Floer cohomology can actually be related to a quantum cohomology which is generated by the quantum ground states of a topological  $\sigma$ -model. Using the more general Novikov ring structure Sadov [9] then argued that these two cohomologies in fact coincide.

Witten's topological nonlinear  $\sigma$ -model [8] is based on solutions of Cauchy-Riemann equations for holomorphic curves

$$\partial_{\sigma}\phi^a + I^a{}_b\partial_{\tau}\phi^b = 0 \tag{12}$$

His approach has led to interesting developments in quantum field theories and string theory [3, 10], but unfortunately from the point of view of Hamiltonian dynamics it corresponds to the non-generic, denegerate special case of (11) with H=0. Consequently it is not clear how the topological  $\sigma$ -model, even if it describes Floer's cohomology, could be applied to understand Arnold's conjecture. For this, one needs to extend the topological  $\sigma$ -model so that it accounts for a *generic* nontrivial Hamiltonian  $H(\phi, \tau)$ , for which the solutions to (2) are non-degenerate.

In the present Letter we shall explain how path integrals and localization techniques, when applied to the topological  $\sigma$ -model, can be used to derive Morse-theoretic relations for classical trajectories in a generic Hamiltonian system. In particular, we shall

explain how the standard, finite dimensional de Rham cohomology relates to quantum cohomology by studying an infinite dimensional version of Poincaré–Hopf and Gauss–Bonnet–Chern formulæ for (11), and by an exact evaluation of the partition function of the topological  $\sigma$ -model we also derive an extension of the Lefschetz fixed point theorem for Floer's instanton equation (10), (11) on the cylinder.

Topological nonlinear  $\sigma$ -model [8] is a theory of maps from a Riemann surface  $\Sigma$  with metric  $\eta_{\alpha\beta}$  and almost complex structure  $\epsilon_{\alpha}{}^{\beta}$  to a manifold X with Riemannian metric  $g_{ab}$  and almost complex structure  $I^a{}_b$ . We assume that the almost complex structures are both compatible with the metrics, so that for example on X we have  $g_{ab} = I^c{}_a I^d{}_b g_{cd}$ . Moreover, if

$$D_c I^a{}_b = \partial_c I^a{}_b + \Gamma^a_{cd} I^d{}_b - \Gamma^d_{cb} I^a{}_d = 0$$
 (13)

 $I^{a}{}_{b}$  is an integrable complex structure and  $g_{ab}$  is Kähler. However, in the following we do not necessarily assume (13).

The fields are the space of maps  $\phi^a: \Sigma \to X$ ,  $a=1\dots \dim X$ . Anticommuting fields  $\psi^a$  are sections of  $\phi^*TX$ , the pullback of the tangent bundle of X. Anticommuting fields  $\rho^a_{\alpha}$ ,  $(\alpha=1,2)$ , and commuting auxiliary fields  $F^a_{\alpha}$  are one-forms on  $\Sigma$  with values on  $\phi^*TX$ , so they are sections of the bundle  $\phi^*TX\otimes T^*\Sigma$ . Let  $\mathcal E$  denote a bundle over the space of maps from  $\Sigma$  to X, whose fibers are sections of  $\phi^*TX\otimes T^*\Sigma$ . Because the rank of  $\mathcal E$  is infinitely bigger than dimension of its base space we must restrict to a sub-bundle, the self-dual part  $\mathcal E^+$ . This means that  $\rho^a_{\alpha}$  and  $F^a_{\alpha}$  both satisfy the self-duality constraint

$$\rho_{\alpha}^{a} = \epsilon_{\alpha}{}^{\beta} I^{a}{}_{b} \rho_{\beta}^{b} \tag{14}$$

The fields have a grading, which at the classical level corresponds to a bosonic symmetry with charges 0, 1, -1, 0 for  $\phi^a, \psi^a, \rho^a_\alpha$  and  $F^a_\alpha$ , respectively.

The action of topological  $\sigma$ -model can be constructed in the following way: Consider a nilpotent operator  $\tilde{Q}$  of degree -1 constructed from the fields of the theory

$$\tilde{Q} = \int_{\Sigma} d^2x \left[ i\psi^a(x) \frac{\delta}{\delta \phi^a(x)} + F_\alpha^a(x) \frac{\delta}{\delta \rho_\alpha^a(x)} \right] \equiv i\psi^a \partial_a + F_\alpha^a \iota_a^\alpha, \tag{15}$$

(In the following summation over a always implies an integration over  $\Sigma$ .) This we identify as a differential operator  $d \otimes 1 + 1 \otimes \delta$  in the superspace defined on the complex  $\Omega(\mathcal{E}) \otimes \Omega(\Pi\mathcal{E})$ . Here  $\Pi\mathcal{E}$  means that the coordinates anticommute. Now introduce a canonical conjugation  $\tilde{Q} \to e^{-\theta}Qe^{\theta}$  so that the cohomologies defined by the operators  $\tilde{Q}$  and Q are the same. A suitable conjugation is defined by

$$\theta = i\psi^c \rho^b_\beta \pi^\alpha_a (\delta_\alpha{}^\beta \Gamma^a_{bc} - \frac{1}{2} \epsilon_\alpha{}^\beta D_c I^a{}_b),$$

where now  $\pi_a^{\alpha} F_{\beta}^b = \delta_{a\beta}^{b\alpha}$ . In a coordinate free language  $\theta = -i(\rho, \hat{\Gamma}\pi)$ , with

$$\hat{\Gamma}_{\alpha b}^{a\beta} = \hat{\Gamma}_{b}^{a} \otimes E_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta} \Gamma^{a}{}_{b} - \frac{1}{2} \epsilon_{\alpha}{}^{\beta} D I^{a}{}_{b}$$

a connection 1-form and  $\psi^a \sim d\phi^a$  denoting the basis of 1-forms on the space of maps  $\Sigma \to X$ . A straightforward calculation gives

$$Q = i(\psi, \partial) + (F + i\rho\hat{\Gamma}, \iota) - i(F\hat{\Gamma}, \pi) - \frac{1}{2}(\rho\hat{R}, \pi).$$

where

$$\frac{1}{2}\hat{R} \ = \ d\hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma}$$

or in components

$$\frac{1}{2}\hat{R}^{a\alpha}_{\beta b} = (\frac{1}{2}R^{a}_{b} - \frac{1}{4}DI^{e}_{b}DI^{a}_{e})\delta_{\beta}^{\alpha} + \frac{1}{4}(I^{a}_{e}R^{e}_{b} - I^{e}_{b}R^{a}_{e})\epsilon_{\beta}^{\alpha}.$$
(16)

which is the Riemann curvature 2-form corresponding to the connection  $\hat{\Gamma}_{\alpha b}^{a\beta}$ . This operator Q is exactly the same as in [8] when we take into account the self-duality condition (14) for  $\rho_{\alpha}^{a}$  and  $F_{\alpha}^{a}$ .

We shall be interested in cohomological actions of the form

$$S = \{Q, \theta\} \tag{17}$$

Such actions are automatically invariant under the BRST-transformation generated by Q and consequently the partition function

$$Z = \int [d\phi^a][dF^a_\alpha][d\psi^a][\rho^a_\alpha] e^{iS}$$
(18)

should remain invariant under arbitrary local variations of  $\theta$ . If we select

$$\theta = (\rho, s) - \frac{\lambda}{4}(\rho, F) = \rho_{\alpha}^{a} g_{ab} \eta^{\alpha\beta} s_{\beta}^{b} - \frac{\lambda}{4} \rho_{\alpha}^{a} g_{ab} \eta^{\alpha\beta} F_{\beta}^{b}, \tag{19}$$

where  $s_{\alpha}^{a}[\phi]$  is a section of  $\mathcal{E}$  and  $\lambda$  is a parameter, we get

$$S = \int_{\Sigma} \left[ -i\rho_{\alpha}^{a} D_{c}(g_{ab}s^{\alpha b}) \psi^{c} + F_{\alpha}^{a} g_{ab}s^{\alpha b} - \frac{i}{2} \epsilon_{\alpha}^{\beta} \rho_{\beta}^{b} D_{c} I^{a}{}_{b} \psi^{c} g_{ad}s^{\alpha d} - \frac{\lambda}{4} F_{\alpha}^{a} F_{a}^{\alpha} \right.$$
$$\left. + \frac{\lambda}{16} D_{c} I^{a}{}_{e} D_{d} I^{e}{}_{b} \psi^{c} \psi^{d} \rho_{a}^{\alpha} \rho_{\alpha}^{b} - \frac{\lambda}{8} R^{a}{}_{bcd} \psi^{c} \psi^{d} \rho_{a}^{\alpha} \rho_{\alpha}^{b} \right]. \tag{20}$$

specializing to  $s_{\alpha}^{a}[\phi] = \partial_{\alpha}\phi^{a}$  and  $\lambda = 1$  then gives the usual action [8] of topological  $\sigma$ -model.

Since the partition function (18) is (formally) invariant under local variations of  $\theta$  we conclude that it must be independent of  $\lambda$ . Indeed, if we eliminate the auxiliary field  $F_{\alpha}^{a}$ , the partition function yields an infinite dimensional version of the Mathai–Quillen formalism [11, 10]:

$$S = \int_{\Sigma} \left[ \frac{1}{4\lambda} (s_{\alpha}^{a} + \epsilon_{\alpha}{}^{\beta} I^{a}{}_{b} s_{\beta}^{b}) (s_{a}^{\alpha} + \epsilon^{\alpha}{}_{\beta} I_{a}{}^{b} s_{b}^{\beta}) - \frac{i}{2} \rho_{a}^{\alpha} D_{c} (s_{\alpha}^{a} + \epsilon_{\alpha}{}^{\beta} I^{a}{}_{b} s_{\beta}^{b}) \psi^{c} - \frac{\lambda}{4} \hat{R}^{a}{}_{bcd} \psi^{c} \psi^{d} \rho_{a}^{\alpha} \rho_{\alpha}^{b} \right]$$

$$(21)$$

the relevant bundle being  $\mathcal{E}^+$  and the section

$$\Phi^a_\alpha = s^a_\alpha + \epsilon_\alpha{}^\beta I^a{}_b s^b_\beta \tag{22}$$

Thus we may view (18) as an infinite dimensional version of the integral of the universal Thom class [10].

Since (18) is independent of  $\lambda$ , we can consider its  $\lambda \to \infty$  limit. For this, we specialize the world-sheet  $\Sigma$  to be a torus with  $\sigma$  and  $\tau$  local coordinates such that the metric  $\eta_{\alpha\beta}$  is a unit matrix with compatible complex structure  $\epsilon_{\sigma}^{\ \tau} = -\epsilon_{\tau}^{\ \sigma} = 1$ . In the  $\lambda \to \infty$  limit we then find that the partition function evaluates to

$$Z_{\lambda \to \infty} = \int [d\phi^a][d\psi^a] \operatorname{Pfaff}(\hat{R}^a{}_b).$$
 (23)

This we identify as the Euler character of the infinite dimensional bundle  $\mathcal{E}^+$ . Formally, this infinite dimensional quantity is a topological invariant and as such does not depend on how we choose the connection. It is the Euler character in the quantum cohomology defined by the quantum ground states of the topological  $\sigma$ -model, and by construction it counts the Witten index *i.e.* the difference in the number of bosonic vacua (even forms) and fermionic vacua (odd forms) in the quantum theory.

In analogy with finite dimensional Morse theory, we next relate the infinite dimensional Euler character (23) to an alternating sum over critical points of a functional  $\Phi$  describing the Floer cohomology. For this we consider the limit  $\lambda \to 0$ , again on a torus  $\Sigma$  with local coordinates  $\sigma, \tau$ .

As  $\lambda \to 0$ , the integral obviously concentrates around the zeroes of (22). For simplicity we shall assume that these zeroes are non-degenerate. (A generalization to the degenerate case is straightforward, see for example [12].) Let  $\phi_0^a$  be such that  $\Phi_\alpha^a[\phi_0] = 0$  and write  $\phi^a = \phi_0^a + \hat{\phi}^a$ . In the absence of degeneracies, the first term in the expansion

$$\Phi^a_{\alpha} \approx \partial_c(\Phi^a_{\alpha})\hat{\phi}^c + \mathcal{O}(\hat{\phi}^2)$$

does not vanish. Using the self-duality of  $\rho_{\tau}^{a}$  and the fact that near  $\phi_{0}$  we have  $\Phi_{\tau}^{a} = -I^{a}{}_{b}\Phi_{\sigma}^{b}$  this gives for the action

$$S = \int_{\tau} \left[ -i\rho_{\sigma}^{a} g_{ab} \partial_{c}(\Phi_{\sigma}^{a}) \psi^{c} + \frac{1}{2\lambda} \partial_{c}(\Phi_{\sigma}^{a}) g_{ab} \partial_{d}(\Phi_{\sigma}^{b}) \hat{\phi}^{c} \hat{\phi}^{d} + \mathcal{O}(\hat{\phi}^{3}) \right]. \tag{24}$$

As  $\lambda \to 0$ , we can then evaluate the partition function which yields

$$Z_{\lambda \to 0} = \int [d\phi_0^a] [d\hat{\phi}^a] [d\psi^a] [\rho_{\tau}^a] [\rho_{\sigma}^a] \det^{-\frac{1}{2}} (\frac{i\lambda}{2}g) \exp[iS]$$

$$= \int [d\phi_0^a] \det^{-\frac{1}{2}} (g) \det^{-\frac{1}{2}} [(\partial_c(\Phi_\sigma^a)g_{ab}\partial_d(\Phi_\sigma^b))] \det(g_{ab}\partial_c\Phi^b\sigma)$$

$$= \sum_{\Phi g = 0} \operatorname{sign} \det ||\partial_b\Phi_\sigma^a||$$
(25)

In particular, if we select  $s^a_{\alpha} = \partial_{\alpha}\phi^a - \chi^a_{\alpha}$  and take  $\chi^a_{\alpha}$  to be a self-dual Hamiltonian vector field, *i.e.* 

$$\chi_{\alpha}^{a} = \frac{1}{2} \partial^{a} H_{\alpha}(\tau, \phi) \tag{26}$$

where  $H_{\alpha}(\phi)$  are two *a priori* arbitrary Hamiltonian functions on X related by the self-duality condition for  $\chi_{\alpha}^{a}$ , we find that the integral localizes to

$$\Phi^a_{\sigma} = \partial_{\sigma} \phi^a + I^a{}_b \partial_{\tau} \phi^b - \partial^a H_{\alpha}(\tau, \phi) = 0 \tag{27}$$

which coincides with Floer's instanton equation (11). Thus we have established that for this equation

$$\sum_{\Phi_{\sigma}^{a}=0} \operatorname{sign} \det ||\partial_{b} \Phi_{\sigma}^{a}|| = \int [d\phi^{a}][d\psi^{a}] \operatorname{Pfaff}(\hat{R}^{a}_{b})$$
 (28)

This is a (formal) infinite dimensional analogue of the familiar Morse theoretic relation between the Gauss-Bonnet-Chern and Poincaré-Hopf formulæ. Note that demanding  $\chi_{\alpha}^{a}$ 's to be self-dual together with (26) implies that  $I^{a}{}_{b}$  must be complex structure so that X is now a Kähler manifold.

In analogy with (9), the present result can be viewed as an infinite dimensional loop space version of the Lefschetz fixed point theorem. For this, we remind that Floer's instanton equation (11) is defined on the cylinder  $S^1 \times R$  with coordinates  $\tau \in [0, T]$  and  $\sigma \in (-\infty, \infty)$ . For a fixed  $\sigma$  the  $\tau$ -dependence defines a closed loop on X, and the  $\sigma$ -dependence parameterizes a smooth flow *i.e.* homotopy mapping of these loops. When  $\sigma \to \pm \infty$  the bounded orbits of (11) tend asymptotically to periodic solutions of (2), implying that we have a flow between periodic classical trajectories only. The additional requirement of periodic boundary condition  $\phi(+\infty,\tau) = \phi(-\infty,\tau)$  then identifies the periodic solutions of (2) with the fixed points of the  $\sigma$ -flow in the loop space. This generalizes the setting of the conventional Lefschetz fixed point theorem that we have discussed in connection of (9) to the loop space. In particular, if we specialize  $H_{\alpha}$  in (28) to coincide with our original Hamiltonian we obtain a lower bound for the number of periodic solutions to (2) in terms of the Witten index of the topological  $\sigma$ -model,

$$\#\{ T - \text{periodic trajectories } \} \ge \left| \int [d\phi^a] [d\psi^a] \operatorname{Pfaff}(\hat{R}^a{}_b) \right|$$
 (29)

The underlying idea in Floer's approach to the Arnold conjecture is that the quantum cohomology of the topological  $\sigma$ -model should coincide with the de Rham cohomology of the original symplectic manifold *i.e.* the target manifold of the  $\sigma$ -model. Such a relation would then provide a natural regularization of the infinite dimensional Euler character in (28), (29) and in particular explains why an estimate such as (3) makes sense as a Morse inequality. We shall now proceed to evaluate our path integral using localization methods to establish that the Euler character (23) of quantum cohomology indeed coincides with the Euler character of the de Rham cohomology over the symplectic manifold X.

For this, we specialize to a symplectic manifold which is Kähler. We select local coordinates so that  $I^a{}_b=i\delta^a{}_b$  and  $I^{\bar a}{}_{\bar b}=-i\delta^{\bar a}{}_{\bar b}$ . Self-duality then implies that  $F^a_z=F^{\bar a}_{\bar z}=0$  so that the only surviving components are  $F^a_{\bar z}$  and  $F^{\bar a}_z$ , and similarly for  $\rho^a_\alpha$ .

Using the (formal) invariance of (18) under local variations of  $\theta$ , we introduce the functional

$$\theta = \eta^{\alpha\beta} g_{ab} F^a_{\alpha} \rho^b_{\beta} + \mu g_{ab} \eta^{\alpha\beta} \partial_{\alpha} \phi^a \rho^b_{\beta} \tag{30}$$

and consider the pertinent action (17). Explicitly (we set  $F_{\bar{z}}^a \equiv F^a \ etc.$ ),

$$S = g_{a\bar{b}}F^aF^{\bar{b}} + R^a{}_{bc\bar{d}}\psi^c\psi^{\bar{d}}\rho^bg_{a\bar{e}}\rho^{\bar{e}} + R^{\bar{a}}{}_{\bar{b}c\bar{d}}\psi^c\psi^{\bar{d}}\rho^{\bar{b}}g_{\bar{a}e}\rho^e + \mu(F^ag_{a\bar{b}}\partial_{\bar{z}}\phi^{\bar{b}} + F^{\bar{a}}g_{\bar{a}b}\partial_{z}\phi^b)$$

$$+ \mu\rho^a(-ig_{a\bar{b}}\partial_{\bar{z}} - ig_{a\bar{d}}\partial_{\bar{z}}\phi^{\bar{e}}\Gamma^{\bar{d}}_{\bar{b}\bar{e}})\psi^{\bar{b}} + \mu\rho^{\bar{a}}(-ig_{\bar{a}b}\partial_{z} - ig_{\bar{a}d}\partial_{z}\phi^e\Gamma^d_{be})\psi^b$$

$$(31)$$

We evaluate the corresponding path integral in the  $\mu \to \infty$  limit, by separating the  $z, \bar{z}$  independent constant modes (for example in a Fourier decomposition) and scale the non-constant modes by  $\frac{1}{\sqrt{\mu}}$ , e.g.

$$\phi^a(z,\bar{z}) = \phi^a_o + \hat{\phi}^a(z,\bar{z}) \rightarrow \phi^a_o + \frac{1}{\sqrt{\mu}}\hat{\phi}^a(z,\bar{z})$$

and similarly for the other fields. Supersymmetry ensures that the Jacobian for this change of variables in (18) is trivial, and evaluating the integrals in the  $\mu \to \infty$  limit using the  $\zeta$ -function regularization we end up with the Euler character of the phase space X in the form

$$Z = \int d\phi_o^a d\phi_o^{\bar{a}} d\psi_o^a d\psi_o^{\bar{a}} \operatorname{Pfaff}(R^a{}_{bc\bar{d}}\psi_o^c \psi_o^{\bar{d}}) \operatorname{Pfaff}(R^{\bar{a}}{}_{\bar{b}c\bar{d}}\psi_o^c \psi_o^{\bar{d}})$$
(32)

which also exhibits the underlying complex structure on X. As a consequence, we have shown that the Euler characters in quantum cohomology and de Rham cohomology coincide and Floer's instanton equation defined over our torus obeys

$$\sum_{\Phi_{\sigma}^{a}=0} \operatorname{sign} \det ||\partial_{c} \Phi_{\sigma}^{b}|| = \sum_{k} (-)^{k} B_{k}$$

with  $B_k$  the Betti numbers of the symplectic manifold X. Obviously this is fully consistent with (3) and relates our infinite dimensional version of the Lefschetz fixed point formula for Floer's equation (28) directly with its finite dimensional counterpart.

In conclusion, we have studied three a priori different cohomologies: Floer's cohomology which describes periodic solutions to Hamilton's equations, Witten's quantum cohomology which describes the quantum ground state structure of a topological nonlinear  $\sigma$ -model, and standard finite dimensional de Rham cohomology. By investigating an infinite dimensional generalization of the familiar Poincaré–Hopf and Gauss–Bonnet–Chern formulæ together with the Lefschetz fixed point theorem, we have shown that these three cohomologies are intimately related. This is consistent with the Arnold conjecture. In particular, our results indicate that topological field theories and functional localization methods appear to be useful tools also in the study of classical dynamical systems.

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